# Wave Reflection in a 2D Ellipse

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**Abstract**

In this paper, we analyze how cutting holes in ellipses affects the power received at one focus following an audio "ping" at the other. We determine that a full ellipse transfers the most power, and analyze why this result occurs. Additionally, we find that power decreases with the square of the percentage of the ellipse removed, and explain the reason behind this relationship. Using our results, one can determine how to most effectively amplify or dampen sound in an ellipsoidal area, and compare it to a simulation of the complex elliptical geometry that makes up the quad of the Franklin W. Olin College of Engineering.

## I. Introduction

The study of acoustics is an important part of architecture and design. Echoes and sound convergence are important considerations in design. The Franklin W. Olin College of Engineering is a good example. "The O" is an oval-shaped quad created by the 3 academic buildings on the campus. The O is notorious for it's acoustic properties. Because of it's shape, sound converges at certain points in the O creating very loud echoes. When designing open spaces like this, it is important to consider the impact the acoustic dynamics the layout will cause. An important question raised by Olin's "O" is how much of an ellipse must be missing to dampen power transmission.

In this paper, we will investigate how this geometry reflects waves emitted from one focus of the ellipse, and how they re-form at the other focus using numeric simulation of the wave equation. We will then look at how removing differently sized sections from the edges of the ellipse changes the properties of the reflected waves, particularly the amount of energy that arrives at the other focus. Based on the results of our simulations of wave propagation in these different geometries, we will see how to maximize and minimize the power transmitted. The maximum power results from a complete ellipse, while the minimum power results from no ellipse at all. This is intuitive, and furthermore we analyze why the drop is power is exponentially related to the percentage of the ellipse removed.

## II. Simulation

### A. Oval Creation

The geometries we tested were ellipses with wedges cut out of the sides of them. We drew the same ellipse for each iteration, and cut out successively wider and wider wedges from the edges, as illustrated in Figure 1. These ellipse drawings were input to our wave propagation simulation as density maps. "Walls" were regions of high density, while air was modeled with low density.

Ultimately, we chose this method of creating density maps in order to easily allow the use of arbitrary geometries. By drawing any grayscale pattern desired in Photoshop, we can easily test wave propagation patterns in any setting.

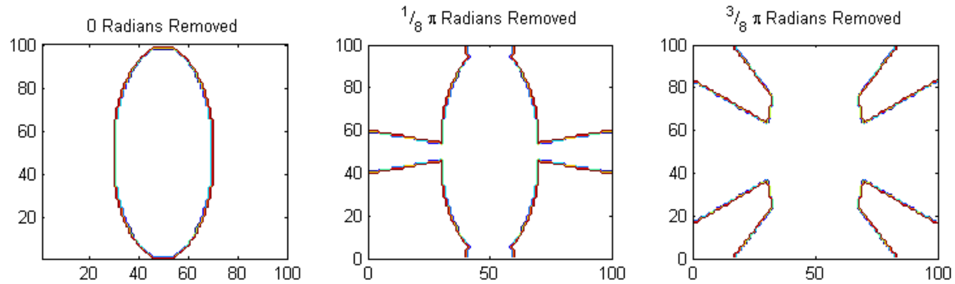



Fig. 1. Oval Geometry Illustration. In our simulation, we tested progressively less-complete ellipses, ranging from a full ellipse to a completely open two dimensional space.

### *B. Wave Equation*

The equation governing the motion of waves through a medium is called the Wave Equation. It relates a state variable's second derivative in time with its second derivative in position, creating an oscillating system in the case of an initial displacement. As an example, the wave equation governing a mass-spring system is given in Equation 1.

$$\frac{\partial^2 u}{\partial t^2} = \frac{k}{\lambda}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

This version of the wave equation is for a physical mass-spring system rather than the pressure-based form of waves that actually make up sound. However, the two are completely analogous and with the correct constants will line up exactly, so this is what we choose to model. In this equation, $k$ is the spring constant of the springs connecting a grid of masses with a density of $\lambda$, and $u$ describes the displacement of each mass element. This partial differential equation is what governs these mechanical waves as the propagate through the system. However, all waves are driven by the same fundamental differential equation with changes only in the constants. In our acoustic system, it is not a wave of position displacement, but a wave of air pressure. $\lambda$ corresponds to the density of the fluid, in our case air, and $k$ is akin to pressure, as it pushes the air molecules apart.

However, in order to implement this into our simulation we need more than a partial differential equation. By manipulating this equation, we can arrive at a discretized solution for the wave equation in two dimensional space that we will be able to model using MATLAB. The solution to the equation is given in Equation 2. In this form of the equation, $\Delta x$ represents the amount of space between simulated points. Additionally, a drag term has now been added to take into account the loss of energy up and down, as well as to friction and other environmental factors. $\beta\dot{u}$ is a viscous drag term that takes energy away from the waves as they propagate.

$$\ddot{u}_{x,y} = \frac{k}{\lambda}\frac{u_{x,y-1} + u_{x,y+1} + u_{x+1,y} + u_{x-1,y} - 4u_{x,y}}{\Delta x^2} - \beta\dot{u}_{x,y} \tag{2}$$

In a perfect situation with no viscous dampening, waves would continue indefinitely, only reflecting when they hit objects. Due to computational limitations, we can not simulate a tremendous number of points, and thus need a way for waves to "die out" when they reach the edge of our simulated area. The technique for this is called "impedance matching", which essentially amounts to adding a viscous drag term to the wave equation that precisely cancels out the wave's motion. In the case of a mass spring system, the correct dampening constant to impedance match is $\sqrt{\frac{T}{\lambda}}$. By setting the $\beta$ term to this in the previous equation, the waves will simply run off the edge of the system (with slight ripples due to numeric error) rather than reflect back off the boundaries.

Using finite analysis in MATLAB, we simulated the propagation of waves through different ellipse geometries. We divided the ellipse geometry into a 100 by 100 grid of points, then used a 4th Order Runge-Kutta approximation to analyze the motion of the wave as time progressed. For each geometry, we "pinged" one of the foci of the ellipse, then recorded the wave amplitudes at the other focus. By integrating the amplitude of the waves at the focus squared (which is proportional to the received power), we measure the amount of energy received at that point.

## III. Results

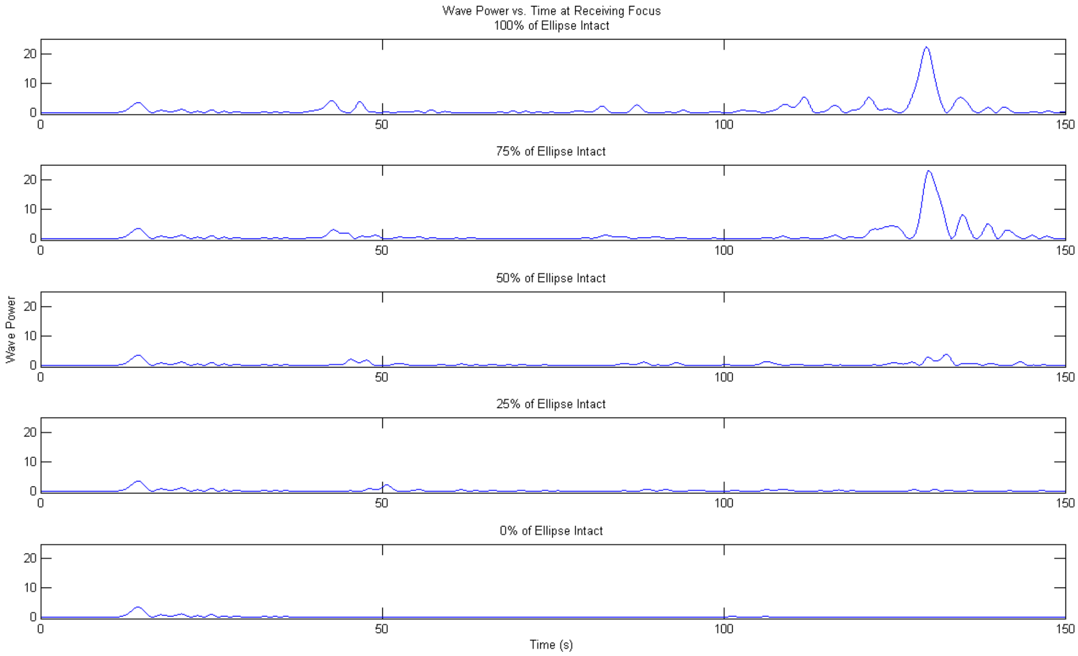
We ran our simulation for 13 different configurations of the ellipse, from a full ellipse with nothing cut out all the way to a flat plane with no ellipse. Our data shows that the amount of energy received decreases with the square of the amount removed, as shown in Figure 2. We found that in testing different ellipses, this relationship remained true.

In waves, power transmitted is proportional to the square of amplitude. Therefore, our definition of power is the square of the amplitude of the waves at the receiving focus. While the units we used don't have any meaning due to the arbitrary constants used in our simulation, the relative values for each cut-out ellipse tell us about what is happening in the system. As time progressed, there were small peaks of waves passing by the receiving focus. The first of these peaks that went by was the same in all the simulations, since this is the first unreflected wave coming straight from the point of origin. This effect is clearly shown in Figure 2. However, as time went on, all of the smaller waves eventually converged into one large peak again, echoing the initial "ping" that triggered them. How tall this peak was had to do with what percentage of the walls were left intact in the system, and closely mirrored the overall power transmitted curve discussed above.

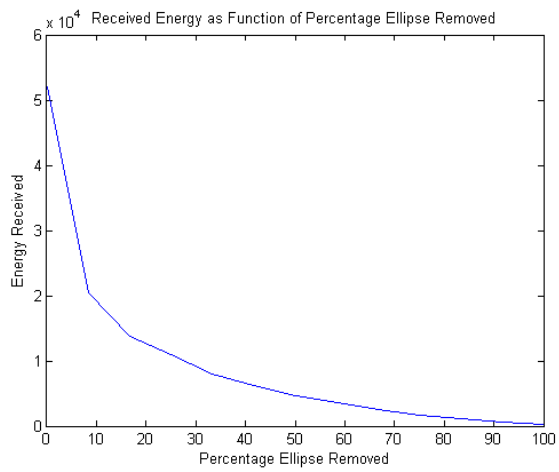
## IV. Analysis

What the data tells us is that, in fact, a solid ellipse is always the most effective way to focus power from a transmission signal placed at one of the foci. However, the energy received at the reception node decreased with the square of the percent of the ellipse removed, not linearly as we expected. However, this is due to our definition of power in the system, which is the square of the amplitudes of the waves. When you remove half the walls, only half of the wave converges, meaning only one quarter of the energy is transmitted. If, rather, we plot the amplitude of the convergence against the amount of the ellipse removed, we see that the amplitude of the convergence decreases linearly as sections of the ellipse are removed.

Our results allow us to determine how to minimize and maximize power transmitted. For the minimum amount, simply do not put up any walls that would reflect sound. The initial pulse passes by, and then the environment




(Power vs. Time)




(Total Energy)

Fig. 2. **Simulation Results. (Top) This figure shows the power received at the reception focus as a function of time. (Bottom) Total Energy Received at Reception Focus as Function Percentage of Ellipse Removed. The energy received at the the reception focus drops as the square of the percentage of the ellipse removed. This makes sense, since power is the wave's amplitude squared, and as we remove so much of the sides, that much less of the wave converges back at the focus.**

flattens out. This is intuitive, and our simulation confirms it. To maximize transmitted power, one should simply leave the entire ellipse intact. Removing sections cause less of the wave to return to converge at the other focus, which directly impacts the amount of energy received.

From this data, we modeled the propagation of waves in "the O" shaped quad on our campus. Using the building plans for the campus, we determined the O has approximately 1/3 of its borders removed. Since it is not a perfect ellipse, we estimated the foci by drawing as close of an ellipse as we could on top. Running this simulation, we found that $6.78\%$ of the transmitted energy was returned at the receiving point. A simulation of a perfect oval returns $14.82\%$ of the energy. The difference probably has to do with our approximation of "foci", and the slightly different geometry of the O. However, this result isn't too far off target, and our model can provide estimations for how oddly shaped geometries would behave.

## V. Conclusion

While we arrived at interesting results through our simulation of the acoustic properties of an ellipse with slices removed, there are many future directions to take in exploring this field. For example, we only tested one ellipse, and we removed sections from it in a very particular way. Other shapes of ellipses and entirely different shapes could be tested. The results we got only analyze a specific set of conditions, and we could expand our simulation out to a more generic case. In addition, it would be interesting to see how moving the transmission node around the ellipse to places other than one focus affects the power received at the other focus.

Another future direction that would help with these goals is further optimizing the simulation. The full set of tests takes over 20 minutes to run, meaning that any small variations are extremely expensive in terms of computational time. If we had a faster simulation, we would be able to test more cases with finer resolution.

Also, presently our simulation only considers two dimensions. Adding a third dimension would allow us to model more complex spaces, and more realistic spaces, such as rooms. Unfortunately, keeping track of a large number of points is very intensive and exponentially slows down the simulation. Our 2D simplification was made to keep things fast and possible without using a supercomputer, but finding a way to expand into 3D would be interesting.

Our simulation helps us understand what occurs with acoustic waves, and why some environments echo and are louder than others. We feel that moving the transmission and reception points around the ellipse and observing the effects on the energy transmitted would be an excellent area to investigate in further studies. This could lead to a greater understanding of how to use geometry to help amplify sound. Ultimately, we found that a full ellipse leads to maximized power transfer, and that having no reflecting walls at all leads to minimized power transfer.

## Acknowledgments

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